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Minimax Theorems for Vector-Valued Multifunctions *

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1 Introduction

We present a Ky Fan type inequality of mixed kind for vector-valued multifunctions. We use it for proving our first type minimax theorem for vector-valued multifunctions. It is a generalization of the classical Sion minimax theorem for scalar functions (in the compact case), as well as, a generalization of a theorem of Tanaka for vector-valued functions.

We use a vector-valued variant for multifunctions of Ky Fan type inequality, described in the another presentation of us in this volume, in order to derive our second type minimax theorems for vector-valued multifunctions, which is stronger than the first one and uses a special notion of convexity for multifunctions.

The theory of vector optimization has been intensively developed in recent years, as currently the interest is focused on vector-valued multifunctions. Important parts of this theory are the minimax problems and saddle point problems, which have their one specific features with respect to the real-valued case. For a development of such vector-valued problems we refer to [T1-T5] and references therein. The vector-valued, set-valued case proposes more possibilities for definitions of saddle points. In this paper we prove also a Nash equilibrium theorem for vector-valued multifunctions using scalarization and Ky Fan's inequality. As a corollary we obtain a loose saddle point theorem for convex-concave multifunctions (with respect to a specified definition). An advantage in our loose saddle point theorems with respect to the existing ones in the literature (see [K-K], [L-V]) is that our conditions are explicit.

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2 Scalar and vector-valued Ky Fan type inequality of mixed kind

Proposition 2. 1 (Scalar Ky Fan type inequality of mixed kind). *Assume that the functions $f, g : K \times K \to \mathbf{R}$, where K is a compact convex nonempty subset of topological vector space, satisfy the properties:*

(i) $f(\cdots, y), g(x, \cdot)$ are lower semicontinuous for every $x, y \in K$;

(ii) $f(x, \cdot), g(\cdot, y)$ are quasi-concave for every $x, y \in K$.

(iii) $\min\{f(x,y), g(x,y)) \leq 0 \quad \forall x, y \in K$.

Then there exist $x_0, y_0 \in K$ such that

$$\min\{\sup_{y\in K} f(x_0, y), \sup_{x\in K} g(x, y_0)\} \leq 0.$$

Proof. Define the function

$$h(\tilde{x}, \tilde{y}, x, y) = \min\{f(\tilde{x}, y), g(x, \tilde{y})\}.$$

It is easy to see that $h(\cdot, \cdot, x, y)$ is lower semicontinuous on $K \times K$ and $h(\tilde{x}, \tilde{y}, \cdot, \cdot)$ is quasiconvex on $K \times K$. Applying the clasical scalar Ky Fan's inequality (see for instance [A-E]), we obtain the result. ■

Let Y be a Banach space, $C \subset Y$ a closed convex cone with nonempty interior and E a topological vector space.

Definition 2. 2 *The multivalued mapping $F : E \to 2^Y$ is called C-properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either*

$$\begin{aligned} F(\lambda x_1 + (1-\lambda)x_2) &\subset F(x_1) - C \qquad or \\ F(\lambda x_1 + (1-\lambda)x_2) &\subset F(x_2) - C. \end{aligned}$$

If $-F$ is C-properly quasiconvex, then F is called C-properly quasiconcave, which is equivalent to $(-C)$-properly quasiconvex mapping.

Definition 2. 3 *We shall say that the multifunction $F : E \to 2^Y$ is C-lower semicontinuous at x_0, if for every $y \in F(x_0)$ and every open $V \ni 0$ there exists an open $U \ni x_0$ such that $(y + V + C) \cap F(x) = \emptyset$ for every $x \in U$.*

Definition 2. 4 *The multifunction F is called C-upper semicontinuous at x_0, if for every $y \in C \cup (-C)$ such that $F(x_0) \subset y + \text{int}C$, there exists an open $U \ni x_0$ such that $F(x) \subset y + \text{int}C$ for every $x \in U$.*

Theorem 2. 5 (Ky Fan type inequality of mixed kind for multifunctions). *Suppose that E_1 and E_2 are topological vector spaces, $X \subset E_1$ is a nonempty convex compact subset, $K \subset E_2$ is a nonempty convex compact subset, C is closed convex strongly pointed cone with nonempty interior in a Banach space Y and $F, G : X \times K \to 2^Y$ are multifunctions satisfying the following conditions:*

(i) $G(x, \cdot)$ is C-quasiconvex for every $x \in X$, and $F(\cdot, y)$ is C-properly quasiconvex for every $y \in K$;

(ii) $G(\cdot, y)$ *is* $-C$*-lower semicontinuous for every* $y \in K$*, and* $F(x, \cdot)$ *is* $-C$*-upper semicontinuous for every* $x \in X$.

(iii) for every $x \in X, y \in K$ *we have: either* $G(x,y) \cap (-\text{int}C) = \emptyset$ *or* $F(x,y) \not\subset -\text{int}C$

.

Then there exist $x_0 \in X, y_0 \in K$ *such that for every* $x \in X, y \in K$ *we have: either* $G(x_0, y) \cap (-\text{int}C) = \emptyset$ *or* $F(x, y_0) \not\subset -\text{int}C$.

Proof. Define

$$\varphi((x,y),(x',y')) := \inf\{f(x,y'), g(x',y))\},$$

where

$$f(x,y) = -\inf_{k\in B}\sup_{z\in F(x,y)} h(k,x,z),$$

$$g(x,y) = -\inf_{k\in B}\inf_{z\in G(x,y)} h(k,x,z)$$

and B is an open base of C. Using Lemmas 3.1, 3.3 of [G-T1] we obtain that $\varphi((\cdot,\cdot),(x',y'))$ is lower semicontinuous for every $x', y' \in K$, and by Lemmas 3.2, 3.4 in [G-T1], $\varphi((x,y),(\cdot,\cdot))$ is quasi-concave for every $x \in X, y \in K$. We have also $\varphi((x,y),(x,y)) \leq 0$ for every $x, y \in K$. Applying Proposition 2.1 we obtain the result. ∎

We shall denote by $\sup A$ (resp. $\inf A$), where $A \subset Y$, the set of all efficient points of the set $\overline{A}$ (the norm closure of A) with respect to C (resp. with respect to $-C$), i.e.

$$\sup A = \{a \in \overline{A} : (a + C) \cap A = \{a\}\};$$

$$\inf A = \{a \in \overline{A} : (a - C) \cap A = \{a\}\}.$$

Recall that A is bounded with respect to C, if the set $(a+C) \cap A$ is bounded for every $a \in A$. A classical lemma of R. Phelps [Ph], which is equivalent to Ekeland's variational principle and which we shall use in the sequel, states that $\sup A \neq \emptyset$ (resp. $\inf A \neq \emptyset$), if A is bounded with respect to C (resp. with respect to $-C$).

We shall say that the multivalued mapping $F : X \to 2^Y$, where X is topological space, is bounded with respect to C, if for every $x \in X$ and every $y \in F(x)$ the set $(y+C) \cap F(x)$ is bounded.

3 Minimax theorems

Theorem 3. 1 (Minimax theorem I). *Suppose that* E_1 *and* E_2 *are topological vector spaces,* $X \subset E_1$ *is nonempty convex compact subset,* $K \subset E_2$ *is a nonempty convex compact subset,* C *is closed convex strongly pointed cone with nonempty interior in a Banach space* Y *and* $F, G : X \times K \to 2^Y$ *are multifunctions, bounded with respect to* C *and* $-C$ *respectively, and satisfying the following conditions:*

(i) $G(x, \cdot)$ *is* C*-quasiconvex for every* $x \in X$*, and* $-F(\cdot, y)$ *is* C*-properly quasiconvex for every* $y \in K$;

(ii) $G(\cdot, y)$ *is* $-C$*-lower semicontinuous for every* $y \in K$*, and* $F(x, \cdot)$ *is* C*-upper semicontinuous for every* $x \in X$.

(iii) for every $x \in X, y \in K$ *and every two vectors* $z_1, z_2 \in Y$ *satisfying* $z_1 - z_2 \notin C$*, we have*

either $[G(x,y) - z_1] \cap (-\mathrm{int}C) = \emptyset$, *or* $z_2 - F(x,y) \not\subset -\mathrm{int}C$.
Then for every z_1 *such that*
(a) $z_1 - \mathrm{int}C \supset \sup \cup_{x\in X} \inf \cup_{y\in K} G(x,y)$,
and for every z_2 *such that*
(b) $z_2 + \mathrm{int}C \supset \inf \cup_{y\in K} \sup \cup_{x\in X} F(x,y)$,
we have $z_1 - z_2 \in C$.

Proof. Assume the contrary. By (ii) it folows that $G(\cdot,y) - z_1$ is $-C$-lower semicontinuous and $z_2 - F(x,\cdot)$ is $-C$-upper semicontinuous. By (i) it follows that $G(x,\cdot) - z_1$ is C-quasiconvex and $z_2 - F(\cdot,y)$ is C-properly quasiconvex. So, using (iii) we apply Theorem 2.5 and obtain that there exist points x_0, y_0 such that for every $x \in X, y \in K$ we have:
either $(G(x_0,y) - z_1) \cap (-intC) = \emptyset$
or $z_2 - F(x,y_0) \not\subset -intC$.
Assume that there exists $x \in X$ such that

$$z_2 - F(x,y_0) \subset -\mathrm{int}C.$$

Then

$$(G(x_0,y) - z_1) \cap (-\mathrm{int}C) = \emptyset \quad \forall y \in K.$$

This implies

$$\Big(\inf \cup_{y\in K} G(x_0,y)\Big) \cap (z_1 - \mathrm{int}C) = \emptyset. \tag{1}$$

It is easy to see, using Phelps lemma (see [Ph]) that for any set S which is bounded with respect to C, we have

$$S \subset \sup S - C \tag{2}$$

So, for $S = \inf \cup_{y\in K} G(x_0,y)$, by (2) we have (using (a))

$$\begin{aligned} \inf \cup_{y\in K} G(x_0,y) &\subset \sup \cup_{x\in X} \inf \cup_{y\in K} G(x,y) - C \\ &\subset z_1 - \mathrm{int}C - C \\ &= z_1 - \mathrm{intC}, \end{aligned}$$

which is a contradiction with (1). Therefore

$$z_2 - F(x,y_0) \not\subset -\mathrm{int}C \quad \forall x \in X.$$

This implies

$$\sup \cup_{x\in X} F(x,y_0) \not\subset z_2 + \mathrm{int}C \tag{3}$$

By (b) and (2) we obtain

$$\begin{aligned} z_2 + \mathrm{int}C &= z_2 + \mathrm{int}C + C \\ &\supset \inf \cup_{y\in K} \sup \cup_{x\in X} F(x,y) + C \\ &\supset \sup \cup_{x\in X} F(x,y_0), \end{aligned}$$

which is a contradiction with (3). ■

Definition 3. 2 *A multifunction* $F : E \to 2^Y$ *is called (in the sense of [K-T-H, Definition 3.6])*

(a) *type-*(v) *C-properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0,1]$ we have either $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1-\lambda)x_2) \subset F(x_2) - C$;*

(b) *type-*(iii) *C-properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0,1]$ we have either $F(x_1) \subset F(\lambda x_1+(1-\lambda)x_2)+C$ or $F(x_2) \subset F(\lambda x_1+(1-\lambda)x_2)+C$.*

If $-F$ is type-(v) [resp. type-(iii)] C-properly quasiconvex, then F is said be type-(v) [resp. type-(iii)] C-properly quasiconcave, which is equivalent to type-(v) [resp. type-(iii)] $(-C)$-properly quasiconvex mapping.

The following theorem is a generalization (in the compact case) of a scalar two-function result of Simon [S, Theorem 1.4], which in turn is a generalization of Sion's minimax theorem [Si].

Theorem 3. 3 (Minimax theorem II). *Suppose that E_1 and E_2 are topological vector spaces, $X \subset E_1$ is a nonempty convex compact subset, $K \subset E_2$ is a nonempty convex compact subset, C is closed convex strongly pointed cone with nonempty interior in a Banach space Y and $F, G : X \times K \to 2^Y$ are multifunctions, bounded with respect to C and $-C$ respectively, such that the set $\cup_{y\in K} \sup \cup_{x\in X} F(x,y)$ is bounded with respect to $-C$ and the set $\cup_{x\in X} \inf \cup_{y\in K} G(x,y)$ is bounded with respect to C. Suppose that F and G satisfy the following conditions:*

(i) $G(x,\cdot)$ is type-(iii) *C-properly quasiconvex on K for every $x \in X$; and $F(\cdot,y)$ is type-*(iii) *C-properly quasiconcave on K for every $y \in K$;*

(ii) $G(\cdot,y)$ is $-C$-lower semicontinuous for every $y \in K$, and $F(x,\cdot)$ is C-lower semicontinuous for every $x \in X$.

(iii) $F(x,y) - G(x,y) \subset -C$ for every $x \in X, y \in K$.

Then there exist two points

$$z_1 \in \sup \cup_{x\in X} \inf \cup_{y\in K} G(x,y)$$

and

$$z_2 \in \inf \cup_{y\in K} \sup \cup_{x\in X} F(x,y)$$

such that $z_1 - z_2 \in C$.

For the proof of this theorem we need the following result.

Theorem 3. 4 ([G-T] Theorem 4.4). *Let K be a nonempty convex subset of a topological vector space E, Y a Banach space, and $F : K \times K \to 2^Y$ a multifunction. Assume that*

1. *$C : K \to 2^Y$ is a multifunction with a closed graph such that $C(x)$ is closed convex cone with compact base $B(x) = (2\overline{B}_Y \setminus B_Y) \cap C(x)$ for every x;*

2. *for every $x, y \in K$, $F(\cdot, y)$ is $C(x)$-lower semicontinuous and locally bounded;*

3. *there exists a multifunction $G : K \times K \to 2^Y$ such that*

   (a) *for every $x \in K, G(x,x) \subset -C(x)$,*

   (b) *$F(x,y) \not\subset -C(x)$ implies $G(x,y) \not\subset -C(x)$,* *(c) $G(x,\cdot)$ is type-(iii) $C(x)$-properly quasiconcave on K for every $x \in K$;*

4. *there exists a nonempty compact convex subset D of K such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x,y) \not\subset -C(x)$.*

Then, the solutions set

$$S = \{x \in K : F(x,y) \subset -C(x), \quad \text{for all } y \in K\}$$

is a nonempty and compact subset of D.

Proof of Theorem 3.3. Define the mapping $H : X \times K \times X \times K \to 2^Y$ by

$$H(\tilde{x},\tilde{y},x,y) = F(x,\tilde{y}) - G(\tilde{x},y).$$

Applying Theorem 3.4 for H we obtain that there exists x_0, y_0 such that

$$H(x_0,y_0,x,y) \subset -C \quad \forall x \in X, \forall y \in K,$$

whence

$$\sup \cup_{x\in X} F(x,y_0) - \inf \cup_{y\in K} G(x_0,y) \subset -C. \tag{4}$$

By (2) we obtain

$$\sup \cup_{x\in X} F(x,y_0) \subset \inf \cup_{y\in K} \sup \cup_{x\in X} F(x,y) + C$$

and

$$\inf \cup_{y\in K} G(x_0,y) \subset \sup \cup_{x\in X} \inf_{y\in K} \cup G(x,y) - C.$$

Therefore, by (4) there exist

$$z_1 \in \sup \cup_{x\in X} \inf_{y\in K} \cup G(x,y), c_1 \in C$$

and

$$z_2 \in \inf \cup_{y\in K} \sup_{x\in X} \cup F(x,y), c_2 \in C$$

such that

$$z_2 + c_2 - (z_1 - c_1) \in -C,$$

which implies

$$z_1 - z_2 \in C + c_1 + c_2 \subset C. \ \blacksquare$$

4 Nash equilibrium and loose saddle point theorems

Definition 4. 1 *The multifunction $F : E \supset X \to 2^Z$, where X is a convex nonempty subset, is called C-convex, if for every $x, y \in X, \lambda \in [0,1], u \in \lambda F(x) + (1-\lambda)F(y)$ there exists $v \in F(\lambda x + (1-\lambda)y)$ such that $u - v \in C$. If F is $-C$-convex, then F is called C-concave.*

Let $k^0 \in intC$ be fixed. Define the functions

$$h(x) = inf\{t \in \mathbf{R} : x \in tk^0 - C\},$$

$$\varphi(x) = \inf h(F(x)),$$

$$\psi(x) = \sup h(F(x)).$$

It is easy to see that h is continuous and sublinear (see [Tam1], [Tam2]).

Lemma 4. 2 *Let the multifunction $F : E \supset X \to 2^Z$ be C-convex. Then the function φ is convex.*

Proof. Let $x_1, x_2 \in X$. By definition of φ and h, for every $\varepsilon > 0$ there exist $z_i \in F(x_i), t_i \in \mathbf{R}, i = 1, 2$ such that

$$z_i - t_i k^0 \in -C \tag{5}$$

and

$$t_i < \varphi(x_i) + \varepsilon.$$

By definition of C-convex multifunction,

$$\exists v \in F(\lambda x_1 + (1-\lambda)x_2) : \lambda z_1 + (1-\lambda)z_2 \in v + C. \tag{6}$$

By (5) we have

$$-C \ni \lambda(z_1 - t_1 k^0) + (1-\lambda)(z_2 - t_2 k^0) = \lambda z_1 + (1-\lambda)z_2 - (\lambda t_1 + (1-\lambda)t_2)k^0. \tag{7}$$

By (6) and (7) we have

$$\begin{aligned} v &\in \lambda z_1 + (1-\lambda)z_2 - C \\ &\subset (\lambda t_1 + (1-\lambda)t_2)k^0 - C - C \\ &= (\lambda t_1 + (1-\lambda)t_2)k^0 - C. \end{aligned}$$

Hence

$$\begin{aligned} h(v) &\leq \lambda t_1 + (1-\lambda)t_2 \\ &< \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) + 2\varepsilon. \end{aligned}$$

Therefore

$$\varphi(\lambda x_1 + (1-\lambda)x_2) := \inf_{z \in F(\lambda x_1 + (1-\lambda)x_2)} h(z) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2). \quad \blacksquare$$

Definition 4. 3 *The multifunction $F : E \to 2^Z$ will be called (C, k^0)-upper semicontinuous at x_0, if for every $\varepsilon > 0$ there exists an open $U \ni x_0$ such that*

$$[(\varphi(x_0) - \varepsilon)k^0 - C] \cap F(x) = \emptyset \quad \forall x \in U.$$

Lemma 4. 4 *If F is $-C$-lower semicontinuous , then φ is upper semicontinuous.*

Proof. Let $x_0 \in E, \varepsilon > 0$ be fixed and $y_0 \in F(x_0)$ be such that

$$h(y_0) < \inf h(F(x_0)) + \varepsilon.$$

By continuity of h, there exists an open $V \ni 0$ such that

$$h(v) < \varepsilon \quad \forall v \in V.$$

By definition of $-C$-lower semicontinuity, there exists an open $U \ni x_0$ such that

$$F(x) \cap (y_0 + V - C) \neq \emptyset \quad \forall x \in U.$$

Let $y \in F(x) \cap (y_0 + V - C)$. Then $y = y_0 + v - c$ for some $v \in V, c \in C$ and we can write

$$\begin{aligned}
\varphi(x) &= \inf_{y' \in F(x)} h(y') \\
&\leq h(y) \\
&\leq h(y_0) + h(v) + h(-c) \qquad \text{(by sublinearity of } h\text{)} \\
&\leq \varphi(x_0) + 2\varepsilon.
\end{aligned}$$

Lemma 4. 5 *If F is (C, k^0)-upper semicontinuous , then φ is lower semicontinuous.*

Proof. Let $x_0 \in E, y \in F(x_0)$ and $x \in U$, where U is given by the definition of (C, k^0)-upper semicontinuity of F at x_0. Let $z \in F(x)$. Then by definition we have:

$$\begin{aligned}
0 &\leq \inf\{t : z - tk^0 \in (\varphi(x_0) - \varepsilon)k^0 - C\} \\
&= \inf\{t : z - (t + \varphi(x_0) - \varepsilon)k^0 \in -C\} \\
&= \varepsilon - \varphi(x_0) + \inf\{t : z - tk^0 \in -C \\
&= \varepsilon - \varphi(x_0) + h(z).
\end{aligned}$$

Hence $\varphi(x_0) \leq h(z) + \varepsilon$, and $z \in F(x)$ is arbitrary, this implies $\varphi(x_0) \leq \varphi(x) + \varepsilon$. ∎

Below we prove a Nash equilibrium type theorem and a loose saddle point theorem. The proofs are based on scalarization via the previous lemmas and on the Ky Fan inequality.

Let E_1, E_2 be topological vector spaces, Z be a Banach space, $X \subset E_1, Y \subset E_2$ be convex compact nonempty subsets and $C_i \subset Z$ be closed convex cones with nonempty interiors, $k_i^0 \in \mathrm{intC}_i, i = 1, 2$.

Theorem 4. 6 **(Nash equilibrium).** *Let the multifunctions $F_i : X \times Y \to 2^Z$ be (C_i, k_i^0)-upper semicontinuous . Assume that $F_1(\cdot, y)$ is C_1-convex for every $y \in Y$, $F_1(x, \cdot)$ is $-C_1$-lower semicontinuous for every $x \in X$, $F_2(x, \cdot)$ is C_2-convex for every $x \in X$ and $F_2(\cdot, y)$ is $-C_2$-lower semicontinuous for every $y \in Y$. Then there exists a Nash equilibrium, $(x_0, y_0) \in X \times Y$, which means*

$$F_1(x, y_0) \cap [\inf h(F_1(x_0, y_0))k_1^0 - \mathrm{int}C_1] = \emptyset \quad \forall x \in X,$$

$$F_2(x_0, y) \cap [\inf h(F_2(x_0, y_0))k_2^0 - \mathrm{int}C_2] = \emptyset \quad \forall y \in Y.$$

Proof. Define

$$f(x, y, \overline{x}, \overline{y}) = \inf h(F_1(x, y)) - \inf h(F_1(\overline{x}, y)) + \inf h(F_2(x, y)) - \inf h(F_2(x, \overline{y}))$$

By Lemma 4.2, $f(x, y, \cdot, \cdot)$ is concave for every $x \in X, y \in Y$ and by Lemmas 4.4, 4.5, $f(\cdot, \cdot, \overline{x}, \overline{y})$ is lower semicontinuous for every $\overline{x} \in X, \overline{y} \in Y$. By Ky Fan's ineqiality (see [A-E, Theorem 6.3.5]) there exists $(x_0, y_0) \in X \times Y$ such that

$$\sup_{(\overline{x}, \overline{y}) \in X \times Y} f(x_0, y_0, \overline{x}, \overline{y}) \leq 0$$

Putting $\overline{y} = y_0$ we obtain

$$\inf h(F_1(x_0, y_0)) \leq \inf h(F_1(x, y_0)) \quad \forall x \in X, \tag{8}$$

and putting $\overline{x} = x_0$ we obtain

$$\inf h(F_2(x_0, y_0)) \leq \inf h(F_2(x_0, y)) \quad \forall y \in Y. \tag{9}$$

But (8) implies

$$F_1(x, y_0) \cap [\inf h(F_1(x_0, y_0))k_1^0 - \mathrm{int}C_1] = \emptyset$$

and (9) implies

$$F_2(x_0, y) \cap [\inf h(F_2(x_0, y_0))k_2^0 - \mathrm{int}C_2] = \emptyset,$$

which finishes the proof. ∎

In the special case when $F_1 = -F_2$ and $C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$, we obtain the following loose saddle point theorem.

Theorem 4. 7 (Loose saddle point theorem). *Suppose that the multifunction $F : X \times Y \to 2^Z$ have compact images and is (C, k^0)-lower semicontinuous and $(-C, -k^0)$-lower semicontinuous, $F(\cdot, y), y \in Y$ is C-convex and C-lower semicontinuous, $F(x, \cdot), x \in X$ is C-concave and $-C$-lower semicontinuous. Then there exists a loose saddle point $(x_0, y_0) \in X \times Y$, namely there exist $z_1, z_2 \in F(x_0, y_0)$, such that*

$$(z_1 - \mathrm{int}C) \cap F(x, y_0) = \emptyset \ \ \forall x \in X,$$

$$(z_2 + \mathrm{int}C) \cap F(x_0, y) = \emptyset \ \ \forall y \in Y.$$

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